Problem Tutorial: "Three Friends (1 point)"

If one of the sentences "`D and J are both liars`" and "`D and Y are both liars`" is true, then it was pronounced by a person who always tells the truth. If these two sentences are false then they were said by two liars. In both cases, there are exactly two liars.

Problem Tutorial: "Triangle Area (1 point)"

The area of the triangles AMP, BNM and CPN is $\frac{3}{16}$ of the area of the triangle ABC, that is 3. Therefore, the area of MNP is $16 - 9 = 7$.

Problem Tutorial: "7-digit Number (1 point)"

There exist $7! = 5040$ 7-digit numbers having each of the digits $1, 2, 3, 4, 5, 6, 7$ exactly once. And there are $6! \cdot 2 = 1440$ such numbers that the digits 1 and 2 are neighbouring, because there are 720 ways to place in some order the digits $3, 4, 5, 6, 7$ and a pair $\{1, 2\}$, and we also need multiply it by 2 because the digit 1 can be placed before the digit 2 or after it. So the answer is $5040 - 1440 = 3600$.

Problem Tutorial: "Multiple of 29 (1 point)"

This number is $100x + 29$ where x is an integer, with sum of digits $29 - 2 - 9 = 18$, so that 9 divides x. And 29 divides $100x + 29$, thus divides $100x$, and then x, since 100 and 29 are coprime. But 9 and 29 are coprime, so that x is a multiple of $9 \cdot 29 = 261$ with the sum of the digits 18. Reciprocally, those conditions are sufficient. The smallest such number is $3 \cdot 261 = 783$.

Problem Tutorial: "Least Possible Difference (1 point)"

We have $|x - w| = 2$ for $x = 2, y = 5, z = 12, w = 0$. On the other hand, $12 = |z - w| = |z - x + x - y + y - w| \leq |y - z| + |x - y| + |x - w| = |x - w| + 10$ and so $2 \leq |x - w|$.

Problem Tutorial: "Regular Polygon (1 point)"

Direct calculations show that $n = 8$ is suitable. If n is even then the largest diagonal is also a diameter of the circumscribed circle, so the only possibility is $n = 8$. If n is odd then we have an isosceles triangle with side lengths 2, 2 and $\sqrt{2 - \sqrt{2}}$. Its smallest angle should be π/n but it is not.

Problem Tutorial: "Find Number (1 point)"

Let this number be $N = 2^a p_1^{s_1} \ldots p_k^{s_k}$, where $p_1, p_2, \ldots, p_k$ are different odd prime numbers. Then N has $(a+1)(s_1+1)\ldots(s_k+1)$ divisors and exactly $(s_1+1)\ldots(s_k+1)$ odd divisors. Thus $(s_1+1)\ldots(s_k+1) = 4$. Obviously the least value of $p_1^{s_1} \ldots p_k^{s_k}$ in this case is $3 \cdot 5 = 15$; and no one can easily obtain $a = 4$ and $N = 2^4 \cdot 3 \cdot 5 = 240$.

Problem Tutorial: "Circle Radius (2 points)"

Solution 1

Let F be a point inside the square such that triangle AFD is equilateral. Notice that BE and AF are parallel and have the same length, then $ABEF$ is a parallelogram and $EF = AB = 6$. Finally, we have $FA = FE = FD = 6$ and F is the circumcenter of AED.

Solution 2

Since $BA = BE$ and $\angle ABE = 150^o$, then $\angle BAE = 15^o$ and $\angle EAD = 75°$. Analogously, $\angle EDA = 75°$. Let P be the circumcenter of AED, we have $PA = PD$ and $\angle APD = 2\angle AED = 60^o$, therefore APD is equilateral and $PA = PD = AD = 6$.

Problem Tutorial: "Maximum Value (2 points)"

Transforming the equality, we will obtain $a^2 + b^2 = 4 - (a + b - 2)^2 \le 4$, and the value 4 is reached by taking $a = 0$, $b = 2$.

Problem Tutorial: "Eleven Segments (2 points)"

If the ordered lengths are $a_1 \le a_2 \le \ldots \le a_{11}$, then a necessary and sufficient condition is that $a_i + a_{i+1} \le a_{i+2}$ for all $i \le 9$. We recognize the Fibonacci sequence, and one computes easily that the minimal value for a_{11} is 89.

Problem Tutorial: "2018 Integers (2 points)"

Note that among two complementary sequences (that is, they are disjoint and their union covers all the written integers), one exactly is positive. There are 2018×2017 sequences that do not cover the whole circle. Half of them are good. And, the sequence covering the whole circle is good, such that the answer is $\frac{2018 \times 2017}{2} + 1 = 2035154$.

Problem Tutorial: "Queens (2 points)"

There cannot be more than 202 queens, since each of 101 rows contains at most two of them. It is possible to place 202 queens (similarly to the board of size 7):

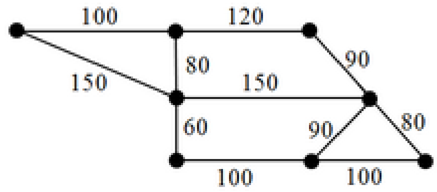
Q	Q					
		Q	Q			
				Q	Q	
Q						Q
	Q	Q				
			Q	Q		
					Q	Q

Problem Tutorial: "Seven Digit Number (2 points)"

Clearly N does not contain 0 as a digit. Since N contains 7 different digits, at least one digit is even, then N is even. If N contains 5 as a digit, then N is divisible by 5 and then N ends in 0 since is even, which is a contradiction. We have to discard one more digit from $\{1, 2, 3, 4, 6, 7, 8, 9\}$, therefore N is divisible by 3 and the sum of its digits is also divisible by 3. Then we have to discard 1, 4 or 7 since $1 + 2 + 3 + 4 + 6 + 7 + 8 + 9 = 40$. Anyway, N is divisible by 9 and we see that we have to discard 4. Finally, the digits of N are $1, 2, 3, 6, 7, 8, 9$ and the sum is 36. One example of such N is 9867312.

Problem Tutorial: "All Streets (2 points)"

The degrees of the vertices A and B in the graph are equal to 3, which is odd (see the figure). If the tourist ends the journey where he started, then the number of times he got in a vertex is equal to the number of times he got out of it. By Euler's theorem, it follows that the tourist has to pass twice through several streets which form a way from A to B. The shortest way from A to B is of length 240, and the total length of all the streets in the city is $1\,120$. So the answer is $240 + 1\,120 = 1\,360$.



Problem Tutorial: "Greatest Prime Divisor (2 points)"

One can prove by induction that $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1$. Thus, $S + 1 = 101!$ and its greatest prime divisor is 101.

Problem Tutorial: "Equilateral Triangles (2 points)"

If one draws x, y and z lines according to 3 orientations with respective angles $\frac{\pi}{3}$, with $x + y + z = 15$, then the number of equilateral triangles is xyz, which is maximized for $x = y = z = 5$. One may check easily that it is not possible to improve this bound by using more than one 'triple orientation'.

Problem Tutorial: "Surjective Functions (3 points)"

A function between finite sets of the same size is surjective if and only if it is bijective. Thus, in the problem, f is a permutation. The condition $f(f(a)) \neq a$ implies, in particular, that $f(a) \neq a$ for all $a \in M$. If we represent the elements of M as the vertices of a graph and draw an arrow from a to $f(a)$, then, since f is a permutation we get that the obtained graph is a union of disjoint (directed) cycles. The conditions $f(a) \neq a$, $f(f(a)) \neq a$ and $f(f(f(a))) \neq a$ mean that f is a 7-cycle. There are exactly $7!/7 = 6! = 720$ distinct 7-cycles.

Problem Tutorial: "Divisible by 83 (3 points)"

Using the general method for solving linear recurrence equations or by induction, one obtains the formula $x_n = 2(3^n - 1)$. Thus, one needs to find the smallest natural k for which $3^k - 1$ is divisible by 83. By Fermat's little theorem $3^{82} - 1$ is divisible by 83, and so k should divide $82 = 2 \cdot 41$. One checks that $3^2 - 1$ is not divisible by 83 but

$$3^{41} - 1 = 3(3^4)^{10} - 1 = 3(81)^{10} - 1 \equiv 3(-2)^{10} - 1 = 3071 = 37 \cdot 83.$$

Problem Tutorial: "Find Distance (3 points)"

The triangles ABP and PCD are similar, then $\frac{AB}{32} = \frac{18}{CD}$, since $AB = CD$, we obtain $AB = CD = 24$. Triangles ABM and CDN are congruent, with $AM = NC$. Moreover, AM and NC are parallel, therefore $AMCN$ is a parallelogram and the midpoint Q of MN is also the midpoint of AC. Then, the distance from Q to the line AD is one half of CD, that is 12.

Problem Tutorial: "Compute the Product (3 points)"

Taking twice the square in each equation, one notes that $a, b, -c$ and $-d$ are roots of the polynomial $x^4 - 14x^2 - x + 43$. The numbers a, b are positive and clearly distinct, and $-c, -d$ are negative and clearly distinct. Therefore $a, b, -c, -d$ are the four distinct roots of the polynomial, hence $abcd = 43$.

Problem Tutorial: "The Greatest Possible Value (3 points)"

Apply AM-GM inequality to the 5 numbers x^5, $\frac{y^5}{4}$, $\frac{y^5}{4}$, $\frac{y^5}{4}$, $\frac{y^5}{4}$, we get $y^5 \leq 2^6$. Equality holds when $x = \sqrt[5]{2^4}$ and $y = \sqrt[5]{2^6}$. Note: The idea of the solution is to apply AM-GM to the $n+1$ numbers $x^5, \frac{y^5}{n}, \ldots, \frac{y^5}{n}$ and then find the suitable n (in order to cancel x).